On the curious commutativity of AMPD matrices

Adhemar Bultheel

Dept. Computer Science, KU Leuven

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http://nalag.cs.kuleuven.be/papers/ade/LEIPZIG18

Definition

$$\textbf{AMPD} = AM + D$$

- $A, D \in \mathbb{C}^{n \times n}$ diagonal matrices
- $M = M_\pi = \overrightarrow{\prod_{k \in \pi}} G_k = G_{\pi_1} G_{\pi_2} \cdots G_{\pi_n},$

  $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ a permutation of $(1, 2, \ldots, n)$
- $$G_k = \begin{bmatrix} I_{k-1} & & & \\ & \alpha_k & \beta_k & \\ & \gamma_k & \delta_k & \\ & & & I_{n-k-1} \end{bmatrix}, \quad k = 1, \ldots, n-1$$

  $$G_n = \begin{bmatrix} I_{n-1} & \\ & \alpha_n \end{bmatrix}.$$
- Note that G_i and G_j if $|i-j| \geq 2$.
- But $G_k G_{k+1} \neq G_{k+1} G_k$ in general.

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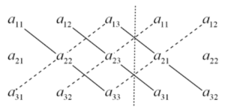
BUT 

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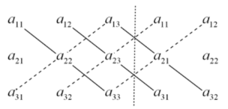

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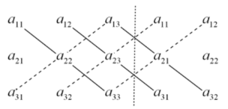
BUT 

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- Is in general $\sigma(AM_\pi + D)$ independent of π?

- Let's do some experiments
- … and the result is….

- Now prove it!!!

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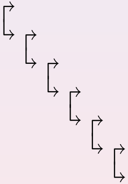
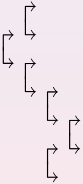
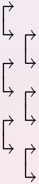
What does M_π look like?

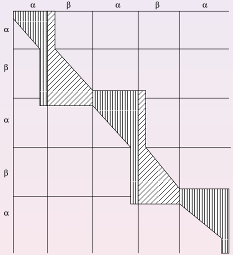
The matrix M is the product of a number of G-matrices

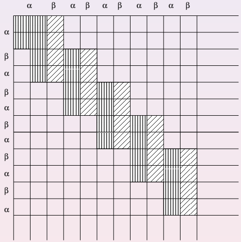
$M_1 =$  or $M_2 =$  or $M_3 =$ 

$M_1 =$ upper Hessenberg matrix $\pi = (1, 2, 3, 4, 5, 6)$
M_1 : $\pi = (2, 1, 3, 4, 6, 5)$ or $(2, 3, 1, 6, 4, 5)$ or ...
$M_3 =$ CMV matrix $\pi = (1, 3, 5, 2, 4, 6)$ or $(5, 1, 3, 4, 2, 6)$, or ...

Shape of the matrix

general order vs. CMV





α upper Hessenberg $\overset{\rightarrow}{\prod_k} G_k$

β lower Hessenberg $\overset{\leftarrow}{\prod_k} G_k$

CMV: alternate $(G_1)(G_3 G_2)(G_5 G_4)\cdots = (G_1 G_3 \cdots)(G_2 G_4 \cdots)$

Cantero-Moral-Velázquez

Proof

Lemma

Let $\mathbb{C}^{n\times n} \ni M' =$ product of $(n-1)$ G-matrices.

$M = \begin{bmatrix} M' & 0 \\ 0 & 1 \end{bmatrix}$ and $G = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{bmatrix}$.

A, D are $(n+1)\times(n+1)$ diagonal matrices.

Then

$$\det(AMG + D) = \det(AGM + D)$$

and hence

$$\sigma(AMG + D) = \sigma(AGM + D).$$

If det are the same then take $D \to D - \lambda I_{n+1}$

This is the characteristic polynomial $\Rightarrow$ the σ are the same.

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- Set $M' = \left[\begin{array}{c|c} M'' & \mathbf{c} \\ \hline \mathbf{r} & m \end{array}\right]$, $A = \mathrm{diag}(A'', a', a)$, $D = \mathrm{diag}(D'', d', d)$.
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$$AGM + D = \left[\begin{array}{c|cc} A''M'' + D'' & A''\mathbf{c} & 0 \\ \hline a'\alpha\mathbf{r} & a'\alpha m + d' & a'\beta \\ a\gamma\mathbf{r} & a\gamma & a\delta + d \end{array}\right]$$

$$AMG + D = \left[\begin{array}{c|cc} A''M'' + D'' & \alpha A''\mathbf{c} & \beta A''\mathbf{c} \\ \hline a'\mathbf{r} & a'\alpha m + d' & a'\beta m \\ 0 & a\gamma & a\delta + d \end{array}\right]$$

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Proof

- work out ...
- regroup ...

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- $\det(AGM + D) = \det(AMG + D) =$
  $[aa' \det G + a'd\alpha] \det \widetilde{M} + (a\delta + d)d' \det(A''M'' + D'')$
  $\widetilde{M} = \mathrm{diag}(A'', 1)M' + \mathrm{diag}(D'', 0)$
- Thus determinants (hence the spectra) are the same.

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Proof

Projection: $P_n^N = [I_n \;\; 0] \in \mathbb{C}^{n \times N}$, (e.g. $P = P_n^{n+1} \Rightarrow M' = PMP^*$)

Theorem

A, D diagonal of size $n+1$
$\pi = (\pi_1, \ldots, \pi_n)$ = permutation of $(1, \ldots, n)$
$M_\pi = G_{\pi_1} G_{\pi_2} \cdots G_{\pi_n} \in \mathbb{C}^{(n+1)\times(n+1)}$, $P = P_n^{n+1}$
Then $\det(P(AM_\pi + D)P^*)$ independent of π
hence also $\sigma(P(AM_\pi + D)P^*)$ independent of π

OK for $n = 2$ (see Example 2)

Proof $n = 2$

Recall Example 2:

$$G_1 G_2 = \begin{bmatrix} \alpha_1 & \beta_1\alpha_2 & \beta_1\beta_2 \\ \gamma_1 & \delta_1\alpha_2 & \delta_1\beta_2 \\ 0 & \gamma_2 & \delta_2 \end{bmatrix} \neq G_2 G_1 = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2\gamma_1 & \alpha_2\delta_1 & \beta_2 \\ \gamma_2\gamma_1 & \gamma_2\delta_1 & \delta_2 \end{bmatrix}$$

$$AG_1 G_2 + D = \left[\begin{array}{cc|c} a_1\alpha_1 + d_1 & a_1\beta_1\alpha_2 & a_1\beta_1\beta_2 \\ a_2\gamma_1 & a_2\delta_1\alpha_2 + d_2 & a_2\delta_1\beta_2 \\ \hline 0 & a_3\gamma_2 & a_3\delta_2 + d_3 \end{array}\right]$$

$$AG_2 G_1 + D = \left[\begin{array}{cc|c} a_1\alpha_1 + d_1 & a_1\beta_1 & 0 \\ a_2\alpha_2\gamma_1 & a_2\alpha_2\delta_1 + d_2 & a_2\beta_2 \\ \hline a_3\gamma_2\gamma_1 & a_3\gamma_2\delta_1 & a_3\delta_2 + d_3 \end{array}\right]$$

hence $\det(P(AG_1 G_2 + D)P^*) = \det(P(AG_2 G_1 + D)P^*)$

Proof Induction step

- Set $\pi' =$ permutation of $(1, \ldots, n-1)$
  Then $M_\pi = G_n M_{\pi'}$ or $M_{\pi'} G_n$.
- Suppose $M_\pi = G_n M_{\pi'}$:

$$P(M_\pi)P^* = P \left[\begin{array}{c|c|c} M'' & \mathbf{c}_{n-1} & 0 \\ \hline \alpha_n \mathbf{r}_{n-1} & \alpha_n m_{n-1} & \beta_n \\ \hline \gamma_n \mathbf{r}_{n-1} & \gamma_n m_{n-1} & \delta_n \end{array}\right] P^* = P \left[\begin{array}{c|c|c} M'' & \mathbf{c}_{n-1} & 0 \\ \hline \alpha_n \mathbf{r}_{n-1} & \alpha_n m_{n-1} & 0 \\ \hline 0 & 0 & 1 \end{array}\right] P^*$$

- $\det(P(AM_\pi + D)P^*) = \det[(PAM_\pi P^*) + PDP^*] = \det(\widehat{A}\widehat{M} + \widehat{D})$
  where $\widehat{A} = \mathrm{diag}(1, \ldots, 1, \alpha_n)PAP^*$, $\widehat{D} = PDP^*$, $\widehat{M} = PM_{\pi'}P^*$
- $\widehat{A}\widehat{M} + \widehat{D} \in \mathbb{C}^{n\times n}$ is an AMPD matrix with $n-1$ G-factors.
- Similarly for $M_\pi = M_{\pi'} G_n$.

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- Similarly for $M_\pi = M_{\pi'} G_n$.

Proof Induction step

- Set $\pi' =$ permutation of $(1, \ldots, n-1)$
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Proof

Corollary

A, D diagonal of size $n+1$
$\pi = (\pi_1, \ldots, \pi_n)$ = permutation of $(1, \ldots, n)$
$M_\pi = G_{\pi_1} G_{\pi_2} \cdots G_{\pi_n} \in \mathbb{C}^{(n+1)\times(n+1)}$
Then $\det(AM_\pi + D)$ independent of π
hence also $\sigma(AM_\pi + D)$ independent of π

Proof: Use $(n+1)$ G-factors with $G_{n+1} = I_{n+2}$.
Apply the previous theorem:
the projection P leaves the product of n G-factors.

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- The eigenvalues are independent of π but the eigenvectors change
- However, suppose all G_k are unitary and do some more experiments.
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- $M_\pi = V_\pi \Lambda V_\pi^*$
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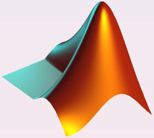
- ... 
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Szegő polynomials

- $\langle f, g \rangle = \int_{\mathbb{T}} \overline{f(z)} g(z) \mu(dz), \quad \mu(\mathbb{T}) = 1$
- $[1, z, z^2, \ldots] \to$ orthonormalize $\to \Phi = [\phi_0, \phi_1, \ldots]$ then

$$z\Phi(z) = \Phi(z)\mathcal{G}, \quad \mathcal{G} = G_0 G_1 G_2 \cdots, \quad G_k = \begin{bmatrix} I_k & & & \\ & -\delta_k & \eta_k & \\ & \eta_k & \overline{\delta}_k & \\ & & & I_\infty \end{bmatrix}$$

  $\delta_k \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\eta_k = \sqrt{1 - |\delta_k|^2} \Rightarrow G_k$ unitary
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- Truncate: $z\Phi_n(z) = \Phi_n(z)\mathcal{G}_n + [0, \ldots, 0, c\phi_{n+1}]$
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- What does this π mean in terms of orthogonal polynomials?
- Szegő OPUC: $[1, z, z^2, z^3, \ldots] \xrightarrow{\perp} [\phi_0, \phi_1, \phi_2, \ldots]$
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- This explains why reordering the unitary G_k factors does not change the absolute values of the entries in eigenvectors.

Szegő polynomials

- What does this π mean in terms of orthogonal polynomials?
- Szegő OPUC: $[1, z, z^2, z^3, \dots] \xrightarrow{\perp} [\phi_0, \phi_1, \phi_2, \dots]$
  CMV OLPUC: $[1, z, z^{-1}, z^2, z^{-2}, z^3, \dots] \xrightarrow{\perp} [\varphi_0, \varphi_1, \varphi_2, \dots]$
  $\varphi_{2k} \in \Pi_{-k,k}, \quad \varphi_{2k+1} \in \Pi_{-k,k+1}, \quad k = 0, 1, 2, \dots$
  and
  $\varphi_{2k}(z) = \varepsilon_{2k}[z^{-k}\phi_{2k}(z)], \quad \varphi_{2k+1}(z) = \varepsilon_{2k+1}[z^{-k}\phi_{2k+1}(z)],$
  $\varepsilon_k \in \mathbb{T}, \quad k = 0, 1, \dots$
- Thus eigenvectors $\tilde{\Phi}_n(z_i) = [\varphi_0, \varphi_1(z_i), \dots, \varphi_n(z_i)]$ satisfy
  $\tilde{\Phi}_n(z_i) = \Phi_n(z_i)E_n, \quad E_n$ a diagonal of constants of modulus 1
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- Mystery solved for M_π matrices.
  But where do the A and D matrices come from?
- OPUC$\to$ORFUC (orthogonal rational functions on $\mathbb{T}$)
  $\alpha_0 = 0, \alpha_1, \alpha_2, \ldots, \quad \alpha_k \in \mathbb{D}$
  $B_0 = 1, B_k(z) = \prod_{i=1}^{k} \frac{z-\alpha_i}{1-\overline{\alpha}_i z}, \quad k = 1, 2, 3, \ldots$
  ORFUC = OPUC if all $\alpha_k = 0$
- $[B_0, B_1, B_2, \ldots] \xrightarrow{\perp} R = [\rho_0, \rho_1, \rho_2, \ldots]$
- OPUC: $\Phi(z) \qquad [z\mathcal{I} - \mathcal{G}] \qquad = 0$

  ORFUC: $R(z)\mathcal{S}^{-1}[(z\mathcal{I} - \mathcal{A}) - (\mathcal{I} - \mathcal{A}^* z)\mathcal{G}'] = 0$

  $\mathcal{A} = \mathrm{diag}(\alpha_0, \alpha_1, \alpha_2, \ldots), \quad \mathcal{S} = \mathcal{S}_\mathcal{A} = (\mathcal{I} - \mathcal{A}^*\mathcal{A})^{1/2}$
  $\mathcal{G}' = G_1' G_2' G_3' \cdots, \quad G_k'$ unitary

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RAMPD

- Can we reorder the factors $G_0 G_1 G_2 \cdots = \mathcal{G}_n$ and keep σ in $\zeta_n(\mathcal{G}_n) := (\mathcal{G}_n + A_n)(I_{n+1} + A_n^* \mathcal{G}_n)^{-1}$?
- Can AMPD be generalized to Rational AMPD (RAMPD)?
- RAMPD: $(AM_\pi + C)(BM_\pi + D)^{-1}, \quad A, B, C, D$ diagonal
- More general: pencils $(AM_\pi + C, BM_\pi + D)$
  generalized eigenvalue problem:
  characteristic polynomial = determinant of $(AM_\pi + C) - (BM_\pi + D)\lambda$
  or $\underbrace{(A - B\lambda)}_{A'} M_\pi + \underbrace{(C - D\lambda)}_{D'} \equiv A' M_\pi + D'$,
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- What does a different π means for ORF?
- Previously: if
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  $R(z)[z\mathcal{I} - \zeta(\mathcal{G})] = 0$ with $\mathcal{G} = G_0 G_1 G_2 \cdots$
- Now: if $\left[B_0, B_1^{-1}, B_2, B_2^{-1}, \ldots\right] \xrightarrow{\perp} \tilde{R} = [\varrho_0, \varrho_1, \varrho_2, \ldots]$ then
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$$B_k = \prod_{i=1}^{k} \frac{z - \alpha_i}{1 - \overline{\alpha}_i z} \quad \text{introduce successive poles } \frac{1}{\overline{\alpha}_i} \notin \mathbb{D}$$

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